A geometric interpretation of the spectral problem for the generalized sine-Gordon system

2001 J. Phys. A: Math. Gen. 34 L153

(http://iopscience.iop.org/0305-4470/34/13/101)

J. Phys. A: Math. Gen. **34** (2001) L153–L159 www.iop.org/Journals/ja PII: S0305-4470(01)17156-8

LETTER TO THE EDITOR

A geometric interpretation of the spectral problem for the generalized sine-Gordon system

Jan L Cieśliński[1] **and Yuriy A Aminov**[2,3]

[1] Uniwersytet w Białymstoku, Instytut Fizyki Teoretycznej, ul. Lipowa 41, 15-424 Białystok, Poland
[2] Uniwersytet w Białymstoku, Instytut Matematyki, ul. Akademicka 2, 15-267 Białystok, Poland
[3] Institute for Low Temperatures of NAS of Ukraine, 47 Lenin Ave., 310164 Kharkov, Ukraine

E-mail: janek@alpha.uwb.edu.pl, janek@fuw.edu.pl, aminov@cksr.ac.bialystok.pl and aminov @ ilt.kharkov.ua

Received 15 September 2000, in final form 14 February 2001

Abstract
The generalized sine-Gordon system is an integrable system implicitly describing submanifolds of negative constant curvature in Euclidean spaces. To obtain the associated spectral problem we consider immersions of these submanifolds in spheres. The spectral parameter turns out to be related to the radius of the spheres. We also derive the so-called Sym–Tafel formula which yields the radius vector of negative constant curvature submanifolds in Euclidean spaces.

PACS numbers: 0230I, 0230J, 0240, 0270H

There are no doubts that spectral problems and, in particular, the spectral parameter play a very important role in the theory of completely integrable systems (see, for instance, [1]). This approach is motivated by the ideas of quantum mechanics: the Korteweg–de Vries equation (the first equation solved by the inverse scattering method) has been associated with the stationary Schrödinger equation [2]. In this case the spectral parameter is related to the energy.

A large class of integrable systems is of geometric origin. In fact one can see a growing overlap of the classical differential geometry of immersions with the modern theory of integrable equations [3–9]. Here we focus our attention on the following system of nonlinear partial differential equations [10, 11]:

$$\begin{aligned}
&\frac{\partial \alpha_{ik}}{\partial x_j} = \alpha_{ij}\beta_{jk} \qquad (k \neq j)\\
&\frac{\partial \beta_{ik}}{\partial x_j} = \beta_{ij}\beta_{jk} \qquad (i, j, k \quad \text{distinct})\\
&\frac{\partial \beta_{jk}}{\partial x_j} + \frac{\partial \beta_{kj}}{\partial x_k} + \sum_{i=1}^{n} \beta_{ij}\beta_{ik} = \alpha_{nj}\alpha_{nk} \qquad (j \neq k)
\end{aligned} \tag{1}$$

where indices i, j, k run from 1 to n ($n \geqslant 2$ is fixed), $(\alpha_{ik}) = A$ is $n \times n$ *orthogonal* matrix, β_{jk} are defined (for $j \neq k$) by the first equation of (1) and, by assumption, $\beta_{kk} = 0$.

The system (1), known as the generalized sine-Gordon system [11], implicitly describes n-dimensional submanifolds of constant sectional curvature $K = -1$ (in other words: Lobachevsky spaces or space forms) immersed in a Riemannian space form of constant curvature $\tilde{K} > -1$ [12, 13]. In particular, it describes immersions of n-dimensional Lobachevsky space L^n of curvature $K = -1$ in the Euclidean space E^{2n-1} and in spheres S^{2n-1} of any radius. This suggests another, perhaps more suitable, name for the system (1), namely an LE system ('Lobachevsky–Euclid') [12].

Note that the coefficients β_{jk} are algebraically expressed by the coefficients of A and their derivatives which means that the dependent variables of the system (1) coincide with variables parametrizing orthogonal matrices of a given dimension n. In the case $n = 2$ the orthogonal matrix A is parametrized by a single angle ω:

$$A = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \tag{2}$$

and the system (1) reduces to the sine-Gordon equation

$$\omega,_{11} - \omega,_{22} = \sin\omega\cos\omega$$

which has numerous applications in physics: crystal dislocations, Josephson junctions, spin waves, Bloch walls in ferromagnetics, field theory models, etc (see, for instance, [14]).

Some physical applications in the case of $n > 2$ have been found as well. For $n = 3$ the system (1) contains as a subsystem the equations describing the dynamics of the rigid body with the fixed point at the centre of mass in the central gravitational field [15]. For $n = 4$ the equations (1) can be interpreted as an analogue of the Maxwell equations for a certain gauge field model [16]. In general, the system (1) is closely related to the Hamiltonian systems with applications in hydrodynamics [17] and topological field theory [18].

In this letter we will show that the integrability of the generalized sine-Gordon system (understood as the existence of the spectral problem and related constructions) is deeply rooted in natural geometrical structures. First, considering immersions of Lobachevsky spaces in the sphere of an arbitrary radius we rediscover the associated spectral problem (we omit the details which can be found elsewhere [19]). The spectral parameter turns out to be expressed in an explicit algebraic way by the radius of the ambient sphere. This type of phenomenon was first noticed by Doliwa and Santini [20, 21] in the case of the evolution of curves in S^3. They derived the Ablowitz–Ladik spectral problems [22] as a consequence of some very simple geometric assumptions.

The second result of this letter, also motivated by results of Doliwa and Santini [20, 21], consists in the derivation of the Sym–Tafel formula for the system (1). The Sym–Tafel formula $F = \Psi^{-1}\Psi,_{\lambda}$ yields some local immersions provided that we have the fundamental solution Ψ of a given spectral problem with the spectral parameter λ [23–25]. For instance, starting from the spectral problem for the sine-Gordon equation we get pseudospherical surfaces [4, 24]. The Bianchi–Bäcklund transformation is reconstructed automatically. In the general case (1) the Sym–Tafel formula has been applied as well [26]. The important advantage of Sym's approach consists in the simplification of the explicit reconstruction of the immersion in the case when the frame consisting of the tangent vectors and normal vectors is given. To find the radius vector of an immersion one has to perform an integration (the problem is solved up to quadratures). The Sym–Tafel formula replaces this integration by differentiation with respect to the spectral parameter. Other important advantages of this formula include the effective discretization procedure [26–28] and unification of integrable nonlinearities [24, 25].

The equations (1) can be obtained as compatibility conditions for a system of linear equations describing the kinematics of the moving frame associated with the immersed submanifold. Indeed, let us consider the general case of an immersion of n-dimensional Lobachevsky space L^n into the sphere $S^{2n-1} \subset E^{2n}$ of radius R. Similarly to the case of an immersion of L^n in a Euclidean space [10, 31], one can prove that there exists n principal directions $\tau_1, \ldots, \tau_n$ (unit vectors which are tangent to the coordinates denoted by $x_1, \ldots, x_n$, respectively) such that the metric and second fundamental forms are simultaneously diagonal. Moreover, the normal bundle is known to be flat which means that in the normal space there exists a basis $\nu_1, \ldots, \nu_n$ such that the corresponding torsion coefficients vanish. Note that one of these normals (say, ν_n) is orthogonal to the sphere S^{2n-1}, i.e. the radius vector of the sphere is given by $R\nu_n$.

Thus the $SO(2n)$-valued function $\Phi := (\tau_1, \tau_2, \ldots, \tau_n, \nu_1, \ldots, \nu_n)^{\mathrm{T}}$ (where the superscript $^{\mathrm{T}}$ means the transposition) explicitly describes the moving frame of tangent and normal vectors associated with the considered immersion. The kinematics of this frame is given by

$$\frac{\partial \Phi}{\partial x_i} = \Omega_i \Phi \tag{3}$$

(for details see [19]) where $\Omega_1, \ldots, \Omega_n$ are $so(2n)$-valued (i.e. traceless and skew-symmetric) matrices of the form

$$\Omega_i := \begin{pmatrix} 0 & \cdots & \beta_{1i} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ -\beta_{1i} & \cdots & 0 & \cdots & -\beta_{ni} & a\alpha_{1i} & \cdots & a\alpha_{n-1,i} & -\frac{1}{R}\alpha_{ni} \\ \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \beta_{ni} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & -a\alpha_{1i} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & -a\alpha_{n-1,i} & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & \frac{1}{R}\alpha_{ni} & \cdots & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where non-zero entries appear only in ith row and ith column, and a is defined by

$$a^2 = 1 + \frac{1}{R^2}. \tag{4}$$

The presented construction yields positive values of the parameters R and a. Parametrizing a and R by λ in the following way:

$$a = \frac{1}{2}\left(\frac{1}{\lambda} + \lambda\right) \qquad \frac{1}{R} = \frac{1}{2}\left(\frac{1}{\lambda} - \lambda\right) \tag{5}$$

we obtain the linear (spectral) problem equivalent (modulo a change of the basis and the signature) to the problem given by Ablowitz *et al* [29] (compare also [26, 30]).

The crucial point is that the compatibility conditions for (3) (given by the LE system (1)) do not depend on R. Therefore the system (3) is a family of linear equations parametrized by R (or, which is more convenient, by λ). The situation is typical for integrable systems. The linear system containing a free parameter (the 'spectral parameter') is known as the linear problem (or Lax pair). The presence of the parameter is important for applying various methods of soliton theory, for example, the Darboux–Bäcklund transformation.

One more remark is in order. Positive a and R correspond to $0 < \lambda < 1$ and only this case results directly from geometry. However, as far as the compatibility conditions are concerned, we may extend the range of λ on the whole real axis (or even on the complex plane). The geometric interpretation of the case $\lambda > 0$ will be discussed below.

The conclusion is clear: the spectral problem with the spectral parameter for the generalized sine-Gordon system coincides with the system of Gauss–Weingarten equations for the local immersion of the Lobachevsky space L^n into the sphere S^{2n-1} of radius R. The spectral parameter λ is related to the radius R by the formula (5).

The linear problem (3) can be rewritten in terms of Clifford numbers (see [26]). To this end we identify generators of the Lie algebra $so(2n)$ with the generators of the Lie algebra of the group $\mathrm{Spin}(2n)$, namely

$$f_{\mu\nu} \leftrightarrow \tfrac{1}{2}\boldsymbol{e}_\mu \boldsymbol{e}_\nu$$

where $f_{\mu\nu}$ are antisymmetric $2n \times 2n$ matrices with coefficients $f_{\mu\nu\alpha\beta} := \delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}$ and $\boldsymbol{e}_\mu$ are generators of the Clifford algebra $\mathcal{C}(2n)$, i.e. they satisfy the relations

$$\boldsymbol{e}_\mu \boldsymbol{e}_\nu + \boldsymbol{e}_\nu \boldsymbol{e}_\mu = 0 \qquad (\mu \neq \nu)$$
$$\boldsymbol{e}_\mu^2 = 1 \quad (\mu = 1, \ldots, 2n).$$

Then, the spectral problem (3) assumes the form

$$\Psi_{,j} = U_j \Psi \tag{6}$$

where Ψ is an element of $\mathrm{Spin}(2n)$ corresponding to the rotation defined by Φ and

$$U_j := \frac{1}{2}\boldsymbol{e}_j \left(-\sum_{k=1}^{n} \beta_{kj}\boldsymbol{e}_k + \frac{\lambda^2+1}{2\lambda}\sum_{i=1}^{n-1}\alpha_{ij}\boldsymbol{e}_{n+i} + \frac{\lambda^2-1}{2\lambda}\alpha_{nj}\boldsymbol{e}_{2n} \right). \tag{7}$$

The compatibility conditions for the linear system (6), (7) coincide, of course, with the generalized sine-Gordon system (1).

Now, we proceed to the geometric derivation of the Sym–Tafel formula. Identifying an appropriate constant orthonormal basis in the ambient space E^{2n} with $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{2n}$ we can express the frame associated with the considered immersion by Ψ, namely $\tau_k = \Psi^{-1}\boldsymbol{e}_k\Psi$, $\nu_k = \Psi^{-1}\boldsymbol{e}_{n+k}\Psi$ $(k = 1, \ldots, n)$. In particular, the radius vector of $L^n \subset S^{2n-1}$ is given by

$$\boldsymbol{r} = R\Psi^{-1}\boldsymbol{e}_{2n}\Psi. \tag{8}$$

The formula (8) has been derived from geometry under the assumptions $R > 0$ and $0 < \lambda < 1$. It can be formally applied also for $\lambda > 1$. Then the formula (5) yields R negative. We are going to explain the geometric meaning of this case. Let us note the following symmetry of the spectral problem (7):

$$U_k(1/\lambda) = \boldsymbol{e}_{2n} U_k(\lambda)\boldsymbol{e}_{2n}.$$

Therefore we can confine ourselves to solutions Ψ satisfying

$$\Psi(1/\lambda) = \boldsymbol{e}_{2n}\Psi(\lambda)\boldsymbol{e}_{2n} \tag{9}$$

which implies that

$$\boldsymbol{r}(1/\lambda) = -\boldsymbol{e}_{2n}\boldsymbol{r}(\lambda)\boldsymbol{e}_{2n}. \tag{10}$$

Geometrically it means that $\boldsymbol{r}(1/\lambda)$ is the reflection of $\boldsymbol{r}(\lambda)$ in the hyperplane orthogonal to $\boldsymbol{e}_{2n}$. The formula (10) can be considered as a geometric definition of $\boldsymbol{r}(\lambda)$ for $\lambda > 1$. The interpretation of negative values of λ can be performed in an analogous way using another symmetry of the spectral problem (7).

The limiting case $\lambda = 1$ formally corresponds to $R = \pm\infty$. To obtain the precise geometric meaning of the limiting process let us consider the formula (10). If $\lambda \to 1-$, then $\boldsymbol{r}(\lambda)$ becomes an immersion in a sphere of the radius $R \to \infty$ and the formula (10) yields the right-hand limit of $\boldsymbol{r}(\lambda)$ at $\lambda = 1$. The immersions $\boldsymbol{r}(1+)$ and $\boldsymbol{r}(1-)$ are symmetric with respect to the hyperplane orthogonal to $\boldsymbol{e}_{2n}$.

The radius vector $\boldsymbol{r}$ is not very convenient because for $R \to \infty$ it tends to infinity. Following [21] (where this idea was applied in the case of integrable evolutions of curves) let us consider

$$\hat{F} := \boldsymbol{r} - R\boldsymbol{e}_{2n} \tag{11}$$

which simply means a change of the frame of reference. Now, the submanifold is described with respect to a fixed point on the sphere (the north pole for $R > 0$ and the south pole for $R < 0$) rather than with respect to the centre of the sphere. Then

$$\hat{F} = R\Psi^{-1}\boldsymbol{e}_{2n}\Psi - R\boldsymbol{e}_{2n} = \frac{2}{\frac{1}{\lambda} - \lambda}\big(\Psi^{-1}\boldsymbol{e}_{2n}\Psi - \boldsymbol{e}_{2n}\big). \tag{12}$$

We can check that

$$\hat{F}\left(\frac{1}{\lambda}\right) = -\boldsymbol{e}_{2n}\hat{F}(\lambda)\boldsymbol{e}_{2n} \tag{13}$$

which means that $\hat{F}(1/\lambda)$ and $\hat{F}(\lambda)$ have identical components orthogonal to the polar axis. Their parallel components are opposite. $\hat{F}(1-)$ yields an immersion in the sphere of infinite radius and can be identified with the immersion into the tangent space to the sphere at the north pole (at least when an immersion of a sufficiently small region is concerned, the global issues can be more complicated). The right-hand limit yields the same immersion reflected in the hyperplane orthogonal to the $\boldsymbol{e}_{2n}$-axis. Thus the vectors $\hat{F}(1+)$ and $\hat{F}(1-)$ belong to tangent spaces at antipodal points. Considering them as elements of the linear space spanned by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{2n-1}$ we have $\hat{F}(1+) = \hat{F}(1-)$ although these vectors are located at different points. In what follows we will compute this limit (shortly denoted by F) assuming that the solution Ψ of the spectral problem (6), (7) is a differentiable function of λ in a neighbourhood of $\lambda = 1$.

Performing the limit $\lambda \to 1$ in the formula (12) we apply L'Hospital's rule to get

$$F := \lim_{\lambda\to 1} \hat{F}(\lambda) = \Psi(1)^{-1}\Psi'(1)\Psi(1)^{-1}\boldsymbol{e}_{2n}\Psi(1) - \Psi(1)^{-1}\boldsymbol{e}_{2n}\Psi'(1) \tag{14}$$

where the prime denotes the derivative with respect to λ. From (9) it follows that

$$\begin{aligned} &\Psi(1)\boldsymbol{e}_{2n} = \boldsymbol{e}_{2n}\Psi(1) \\ &\boldsymbol{e}_{2n}\Psi'(1) = -\Psi'(1)\boldsymbol{e}_{2n}. \end{aligned}$$

Therefore from (14) we get immediately the Sym–Tafel formula [4, 24]:

$$F = 2\Psi(1)^{-1}\Psi'(1)\boldsymbol{e}_{2n} = -2\boldsymbol{e}_{2n}\ \Psi^{-1}\frac{\partial\Psi}{\partial\lambda}\bigg|_{\lambda=1}. \tag{15}$$

The factor $\boldsymbol{e}_{2n}$ just projects $\Psi^{-1}\Psi_{,\lambda}$ (assuming values in $E^{2n} \wedge \boldsymbol{e}_{2n} \simeq E^{2n-1}$) onto E^{2n-1}.

The Sym–Tafel formula has been successfully applied to several classes of surfaces in E^3 (compare [4, 5, 8, 25] and references quoted therein). It would be interesting to find a geometrical derivation of the Sym–Tafel formula in the case of $K = \text{const}$ surfaces, $H = \text{const}$ surfaces, isothermic surfaces, Bianchi surfaces and other integrable classes of immersions.

The results presented in this letter are included with more details in a larger paper [19]. One can also find there other new results on immersions of Lobachevsky spaces. We discuss special submanifolds generalizing Clifford tori and differential equations describing the Bäcklund transformation in the case of $L^2 \subset S^3$. In particular, we study the regularity of the Bäcklund transformation image.

This research was supported, in part, by the KBN grant 2 P03B 143 15.

References

[1] Zakharov V E, Manakov S V, Novikov S P and Pitaievsky L P 1980 *Theory of Solitons* (Moscow: Nauka)

[2] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 Method for solving the Korteweg–de Vries equation *Phys. Rev. Lett.* **19** 1095–7

[3] Chern S S 1991 Surface theory with Darboux and Bianchi *Miscellanea Mathematica* ed P Hilton, F Hirzebruch and R Remmert (Berlin: Springer) pp 59–69

[4] Sym A 1985 Soliton surfaces and their application. Soliton geometry from spectral problems *Geometric Aspects of the Einstein Equations and Integrable Systems (Lecture Notes in Physics vol 239)* ed R Martini (Berlin: Springer) pp 154–231

[5] Bobenko A I 1994 Surfaces in terms of 2 by 2 matrices. Old and new integrable cases *Harmonic Maps and Integrable Systems (Aspects of Mathematics vol 23)* ed A P Fordy and J C Wood (Brunswick: Vieweg)

[6] Hu H S 1995 Solitons and differential geometry *Soliton Theory and Its Applications* (Berlin: Springer) pp 297–336

[7] Konopelchenko B G 1996 Induced surfaces and their integrable dynamics *Stud. Appl. Math.* **96** 9–51

[8] D Wójcik and J Cieśliński (ed) 1998 *Nonlinearity & Geometry* (Warsaw: Polish Scientific Publishers) Especially see the paper: J Cieśliński The Darboux–Bianchi–Bäcklund transformation and soliton surfaces 81–107

[9] Terng C L and Uhlenbeck K 2000 Geometry of Solitons *Not. Am. Math. Soc.* **47** 17–25

[10] Aminov Yu A 1977 On immersions of regions of the n-dimensional Lobachevsky space into $(2n-1)$-dimensional Euclidean space *Dokl. Akad. Nauk SSSR* **236** 521–4 (in Russian) (Engl. Transl. 1977 *Sov. Math. Dokl.* **18** 1210–3)

[11] Tenenblat K and Terng C L 1979 A higher-dimension generalization of the sine-Gordon equation and its Bäcklund transformation *Bul. Am. Math. Soc. (N.S.)* **1** 589–93

[12] Aminov Yu A 1998 *New Ideas in Differential Geometry of Submanifolds* (Kharkov: Acta Academic)

[13] Tenenblat K 1998 *Transformations of Manifolds and Applications to Differential Equations (Pitman Monographs and Surveys in Pure and Applied Mathematics 93)* (Harlow: Addison-Wesley Longman)

[14] Barone A, Esposito F, Magee C J and Scott A C 1971 Theory and applications of the sine-Gordon equation *Riv. Nuovo Cimento* **1** 227–67

[15] Aminov Yu A 1982 Multidimensional analogue of the 'sine-Gordon' equation and the motion of a rigid body *Dokl. Akad. Nauk SSSR* **264** 1113–16 (in Russian) (Engl. Transl. 1982 *Sov. Phys. Dokl.* **27**)

[16] Aminov Yu A 1988 Isometric immersions of domains of the n-dimensional Lobachevsky space into Euclidean spaces with flat normal connection. A gauge field model *Mat. Sbornik* **137 (179)** 275–99 (in Russian) (Engl. Transl. 1990 *Math. USSR Sbornik* **65** 279–303)

[17] Tzarev S P 1992 Classical differential geometry and integrability of systems of hydrodynamic type *Applications of Analytic and Geometric Methods to Nonlinear Differential Equations* ed P A Clarkson (*NATO ASI Series*) (Dordrecht: Kluwer) pp 241–9

[18] Dubrovin B 1992 Integrable systems in topological field theory *Nucl. Phys.* B **379** 627–89

[19] Aminov Yu A and Cieśliński J L 2000 The immersions of regions of Lobachevsky spaces into spheres and Euclidean spaces and geometric interpretation of the spectral parameter *Inst. Math. University Białystok preprint* No IM UwB/01/2000

[20] Doliwa A and Santini P 1994 The integrable dynamics of discrete and continuous curves *Inst. Theor. Phys. Warsaw University preprint* No IFT/22/94 Warsaw

[21] Doliwa A and Santini P 1996 The integrable dynamics of a discrete curve *Symmetries and Integrability of Difference Equations* (*CRM Proc. and Lecture Notes vol 9*) ed D Levi, L Vinet and P Winternitz (Providence, RI: American Mathematical Society) pp 91–102

[22] Ablowitz M J and Ladik J F 1975 Nonlinear differential-difference equations *J. Math. Phys.* **16** 598

[23] Sym A 1982 *Soliton Surfaces Lett. Nuovo Cimento* **33** 394–400

[24] Sym A 1983 Soliton surfaces II. Geometric unification of solvable nonlinearities *Lett. Nuovo Cimento* **36** 307–12

[25] Cieśliński J 1997 A generalized formula for integrable classes of surfaces in Lie algebras *J. Math. Phys.* **38** 4255–72

[26] Cieśliński J 1997 The spectral interpretaton of n-spaces of constant negative curvature immersed in R^{2n-1} *Phys. Lett.* A **236** 425–30
[27] Bobenko A and Pinkall U 1996 Discrete surfaces with constant negative Gaussian curvature and the Hirota equation *J. Diff. Geom.* **43** 527–611
[28] Bobenko A and Pinkall U 1996 Discrete isothermic surfaces *J. Reine Angew. Math.* **475** 187–208
[29] Ablowitz M J, Beals R and Tenenblat K 1986 On the solution of the generalized wave and generalized sine-Gordon equations *Stud. Appl. Math.* **74** 177–203
[30] Ferus D and Pedit F 1996 Isometric immersions of space forms and soliton theory *Math. Ann.* **305** 329–42
[31] Moore J D 1972 Isometric immersions of space forms in space forms *Pac. J. Math.* **40** 157–66